The UIL Math Sense contest is all about doing math in your head. That means that there are quite a few tricks you have to know to figure out the tough questions. Some of them require that you simplify the problem to a simpler problem that you know how to do. We will try to cover some of the basic tricks and show you how they work too.

Remember these key ideas for the UIL test:

- Do the problems quickly in your head...
- Do the easiest problems first...
- Look for patterns that make problems easier...
- If you get stuck, move on to the next problem...
1 All About 10’s

The *UIL Number Sense* stresses computing by reducing things to easy numbers you can do in your head.

First, they do a lot of two-by-two multiplication that can be done by breaking down one of the terms to 10 plus something: $15 \times 25$. You could use the cross method, but its slower than breaking the problem down in your head:

$$15 \times 25 = (10 + 5) \times 25 = 250 + 125 = 375$$

Try these practice examples:

1. $13 \times 50 =$
2. $46 \times 101 = 46 \times (100 + 1) =$
3. $12 \times 75 =$
4. $42 \times 15 = 42 \times (10 + 10/2) =$
5. $45 \frac{1}{2} \text{ hours} = \underline{\text{__________}}$ (approximate minutes)
   \[
   \frac{45}{45} \times 60 = 45 \times 6 \times 10 = \underline{\text{__________}}
   \]
   (forget the 1/3 hour minutes.. Remember, this is approximate!).
6. $874 \times 240 = \underline{\text{__________}}$ (approximate)
   \[
   \frac{874}{900} \times 240 = 9 \times 24 \times 1000 \text{ (Remember, this is approximate!).}
   \]
7. $2011 \times 51 = \underline{\text{__________}}$ (approximate)
   \[
   \frac{2011}{2000} \times 50 = 2 \times 5 \times 1000 \text{ (Remember, this is approximate!).}
   \]
8. $5 \times 6 \times 7 =$
   (Hint: multiply 5 on 6 first since it gives a multiple of 10)
9. $(-5)(6)(-7) =$
   (Hint: Negatives cancel, use previous problem. This is a real UIL problem!)
10. $8^2 + 24^2 =$
    (Hint: decompose $24^2$ into $3^2 \times 8^2$, then you’ll have 10 * Something.)
The Cross Method is faster than many of the other tricks they teach in the UIL trick book and you only need to know the one method for most of the problems.

In the cross method, you get the middle number by multiplying the "cross terms" and add them (with carry-over) instead of writing down two lines of numbers and adding. Its faster. When we say "cross terms" we mean:

Cross Terms = (top left)*(bottom right) + (top right)*(bottom left)

So the digits you write down are (from right to left)
1. Multiply the two right-most digits, write down the 1’s (carry-over 10’s if needed)
2. Multiply the cross, write down the 1’s (carry-over if 10’s needed)
3. Multiply the two left-most digits, write down the entire number

![Figure 1: The Cross Method](image-url)
Here is an example of the traditional way:

\[
\begin{array}{c}
16 \\
\times 21 \\
\hline
16 \\
+ 320 \\
\hline
336 \\
\end{array}
\]

Here is the cross method

\[
\begin{array}{c}
16 \\
\times 21 \\
\hline
336 \\
\end{array}
\]

You start from the right and work left: To get the result you first multiply the 1 and 6, and write 6. Then you get the cross by multiplying 1x1 plus 2x6 = 13, you write 3 and carry 1. Next you multiply the 2 ends 2x1 and add the carry-over 1 to get 3 which is the last digit.

Here is another Cross method example for a larger number:

\[
\begin{array}{c}
26 \\
\times 42 \\
\hline
1092 \\
\end{array}
\]

You start from the right and work left: First digit: multiply the 2 and 6, and write 2, carry 1. Second digit: cross multiply \(2 \times 2 + 4 \times 6 = 28\) add carryover (1) to get 29: write 9 and carry 2. Last digits: \(4 \times 2\) plus carryover (2) gives 10.
In number sense you are not allowed to write down ANYTHING so you need a method that can be done in your head. It takes about half the time to use the cross Method than the traditional one too.

Here are some examples to try. Don’t forget that you may have to carry-over at each step...

\[
\begin{array}{cccc}
12 & 24 & 54 & 45 \\
\times 13 & \times 33 & \times 27 & \times 95 \\
\end{array}
\]

\[
\begin{array}{cccc}
21 & 34 & 57 & 83 \\
\times 11 & \times 11 & \times 11 & \times 11 \\
\end{array}
\]

\[
\begin{array}{cccc}
12 & 24 & 54 & 45 \\
\times 33 & \times 44 & \times 55 & \times 88 \\
\end{array}
\]

\[
\begin{array}{cccc}
53 & 67 & 34 & 85 \\
\times 27 & \times 13 & \times 32 & \times 91 \\
\end{array}
\]

\[
\begin{array}{cccc}
13 & 27 & 34 & 45 \\
\times 17 & \times 13 & \times 22 & \times 31 \\
\end{array}
\]
3 Double Cross Method Multiplying 3-on-2: (Elevens)

The *cross method* we just covered can be used to multiply 3 on 2 numbers. But you need to do the cross twice: the 2nd and 3rd digits instead of just the middle one. Don’t for get to carry-over each time. Here is an example of how it works:

\[
136 \\
\times \quad 21 \\
\hline
2856
\]

(First cross is \(21 \otimes 36\) and second is \(21 \otimes 13\))

1. The first digit is just the right-most ends multiplied : write 6
2. The second digit is first cross: \(1 \times 3 + 2 \times 6 = 15\). Write 5 carry 1
3. The third digit is second cross: \((1 \times 1 + 2 \times 3) + 1\). Write 8
4. The last digit (left-most) just \(2 \times 1\), write 2

The UIL only gives three-on-two when multiplying by 11, so you really just add the two top digits: Let’s try some examples:

\[
\begin{array}{cccc}
213 & 234 & 534 & 145 \\
\times & 11 & \times & 11 \\
\hline
\end{array}
\]

\[
\begin{array}{cccc}
543 & 267 & 394 & 835 \\
\times & 21 & \times & 13 \\
\hline
\end{array}
\]

\[
\begin{array}{cccc}
133 & 257 & 394 & 143 \\
\times & 12 & \times & 17 \\
\hline
\end{array}
\]

The last one is on the UIL sample test.
4 The Power of Powers

When you raise a number to a power, the number is multiplied to itself that many times. When we say “A to the power n” we write it as $A^n$ and it means you take n copies of A and multiply them together. For example $10^2 = 10 \times 10$ and $10^3 = 10 \times 10 \times 10$. Powers can save you a lot of time and are used all throughout science and engineering. Next we talk about the rules of powers.

4.1 Power Laws: $A \neq 0, B \neq 0$

\[
A^0 = 1 \quad ; \quad A^1 = A \quad (4-1)
\]
\[
A^n \times A^m = A^{n+m} \quad (4-2)
\]
\[
\frac{A^n}{A^m} = A^{n-m} \quad (4-3)
\]
\[
A^n \times B^n = (A \times B)^n \quad (4-4)
\]
\[
A^n / B^n = (A / B)^n \quad (4-5)
\]
\[
A^{-n} = 1/A^n \quad ; \quad 1/A^{-n} = A^n \quad (4-6)
\]

4.2 Examples:

- $10^0 = 1$
- $10^1 = 10$
- $10^2 = 10 \times 10$
- $10^3 = 10 \times 10 \times 10$
- $10^3 \times 10^3 = 10^6$ (one million)
- $10^3 \times 10^6 = 10^{12}$ (one billion)
- $10^3 \times 10^{12} = 10^{15}$ (one trillion)
- $2^8 / 2^2 = 2^6$ Rule (4-3)
- $12^2 \times 36^2 = 12^2 + (3 \times 12)^2 = 12^2 + (3^2 \times 12^2) = (1 + 3^2) \times 12^2 = 1440$
- $6 \times 10^3 + 5 \times 10^2 + 3 \times 10^0 = 6503$
- $10^3 \div 5^3 = (\frac{10}{5})^3 = 2^3 = 8$
- $2^3 \div 8^3 = (\frac{2}{8})^3 = (\frac{1}{4})^3 = \frac{1}{64}$
4.3 Exercises

Write the following as a regular decimal number:

Ex 4.1: \( 2^0 = \)
Ex 4.2: \( 2^1 = \)
Ex 4.3: \( 2^2 = \)
Ex 4.4: \( 2^3 = \)
Ex 4.5: \( 2^4 = \)
Ex 4.6: \( 2^5 = \)
Ex 4.7: \( 2^6 = \)
Ex 4.8: \( 10^0 = \)
Ex 4.9: \( 10^1 = \)
Ex 4.10: \( 2 \times 10^3 + 3 \times 10^1 + 4 \times 10^0 + = \)
Ex 4.11: \( 4 \times 10^4 + 8 \times 10^3 + 7 \times 10^1 + 2 \times 10^0 + = \)

Write the following as fractions:

Ex 4.12: \( 10^{-1} = \)
Ex 4.13: \( 10^{-2} = \)
Ex 4.14: \( 10^{-3} = \)
Ex 4.15: \( 3^{-2} = \)
Ex 4.16: \( 3^{-3} = \)
Ex 4.17: \( 2^{-1} = \)
Ex 4.18: \( 2^{-2} = \)
Ex 4.19: \( 2^{-3} = \)
Ex 4.20: \( 2^{-4} = \)
Ex 4.21: \( 5^{-2} = \)
Ex 4.22: \( 6^{-2} = \)
Ex 4.23: \( 2^{-1} + 2^{-2} = \)
Write the following as a power:

**Ex 4.24:** \(2^3 \div 2^7 = \)

**Ex 4.25:** \(3^2 \times 2^3 = \)

**Ex 4.26:** \(2^{-1} - 2^{-2} = \)

**Ex 4.27:** \(4^{-1} - 2^{-3} = \)

**Ex 4.28:** \(10 \times 10^3 = \)
5 Adding Sequences

5.1 Re-Balancing Numbers in Sums

Rebalancing means that you notice that you can make all numbers in a sum the same. In most cases you want to look at the middle number for a clue.

For example we can make all the numbers in $7 + 8 + 9$ to be just $8 + 8 + 8$ by moving one from the 9 and giving it to 7. This becomes a very important tool when you have a problem like this:

$$26 + 28 + 30 = 28 + 28 + 28 = 28 \times 3 = 60 + 24 = 84$$

Notice that in the last step, I decomposed the multiplication by 10:

$$3 \times 28 = 3 \times (20 + 8) = 3 \times 20 + 3 \times 8$$

Ok, here is an example almost exactly like a UIL questions:

$$27 + 29 + 31 = 29 + 29 + 29 = 90 - 3 = 87$$

Notice that it is faster and easier to do $3 \times 29 = 3 \times (30 - 1)$ than to do $3 \times 20 + 3 \times 9$, because I get to use 10’s. Here is another one where you need to re-balance more than 2:

$$32 + 36 + 40 = 36 + 36 + 36 = 90 + 18 = 108$$

Here are some exercises:

**Ex 5.29:** $19 + 21 + 23 = $
**Ex 5.30:** $19 + 21 + 23 + 25 + 27 = $
**Ex 5.31:** $30 + 31 + 32 + 33 + 34 = $
**Ex 5.32:** $49 + 51 + 53 + 55 = $
**Ex 5.33:** $11 + 15 + 19 + 23 + 27 = $
**Ex 5.34:** $15 + 19 + 23 + 27 + 31 = $
**Ex 5.35:** $1 + 2 + 3 + 4 + 5 + 6 + 7 = $
When asked to add all the whole numbers between 1 and 100, Gauss told his teacher the answer in only a few seconds: 5050. He did this by folding (re-balancing ?) the numbers at 50 and adding:

\[
\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & \cdots & 97 & 98 & 99 & 100 \\
\end{array}
\]

Now what if we folded the numbers over at the half way point:

\[
\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & \cdots & 47 & 48 & 49 & 50 \\
+ & 100 & 99 & 98 & 97 & 96 & \cdots & 54 & 53 & 52 & 51 \\
\end{array}
\]

\[
\begin{array}{ccccccccc}
\end{array}
\]

Add 50 terms: \(50 \times 101 = 5050\)

But we still need a formula. Lets take a look at calculation as a function of the biggest number \(N = 100\). First notice that we have 50 terms to add, which is \(N/2\). Next notice that the number 101 is just \(N + 1\). The formula that comes out of this is:

\[
S(N) = \frac{N}{2} \times (N + 1) = \frac{N(N + 1)}{2}
\]

We often use the Greek symbol \(\sum\) (capital sigma) in mathematics to denote the sum of many things. Here is how the formula looks in the fancy math journals:

\[
S(N) = \sum_{n=1}^{N} n = \frac{N}{2} \times (N + 1) = \frac{N(N + 1)}{2}
\]

The numbers on the top and bottom of \(\sum\) indicate that you add all the numbers from 1 to \(N\). Test this formula for \(N = 5\), \(N = 10\), \(N = 100\), and \(N = 1000\).
5.3 Adding Odd Numbers

The UIL sometimes asks that you add numbers like this:

\[ 1 + 3 + \cdots + 17 + 19 \]

Its easier to use a formula than to add all those numbers in your head. The formula we get is similar to the Gauss trick. We fold the numbers over and then add:

\[
\begin{array}{cccccc}
 & 1 & 3 & 5 & 7 & 9 \\
+ & 19 & 17 & 15 & 13 & 11 \\
= & 20 & 20 & 20 & 20 & 20 \\
\end{array}
\]

Add 5 terms: \(5 \times 20 = 100\)

But what is the formula? Well, we started with only 10 terms, which is \((N + 1)/2\) when \(N = 19\). And when we folded it over we have only 5 terms which is \((N + 1)/4\) terms. Each term is worth 20 which is \(N + 1\) so the formula is:

\[ S_{\text{odd}}(N) = \frac{N + 1}{4} \times (N + 1) = \frac{(N + 1)^2}{4} = \left(\frac{N + 1}{2}\right)^2 \]

Test this formula with \(N = 19\), \(N = 21\), and \(N = 51\).
5.4 Adding Even Numbers

The UIL sometimes asks that you add sequential even numbers:

\[ 2 + 4 + \cdots + 18 + 20 \]

The formula we get is similar to the other ones. We fold the numbers over and then add:

\[
\begin{array}{c}
2 \\
4 \\
6 \\
8 \\
10 \\
+ \\
20 \\
18 \\
16 \\
14 \\
12 \\
= \\
22 \\
22 \\
22 \\
22 \\
22
\end{array}
\]

Add 5 terms: \(5 \times 22 = 110\)

Now what is the formula? Well, we started with only 10 terms, which is \(N/2\) when \(N = 20\). And when we folded it over we have only 5 terms which is \(N/4\) terms. Each term is worth 22 which is \(N + 2\) so the formula is:

\[ S_{\text{even}}(N) = \frac{N}{4} \times (N + 2) = \frac{N \times (N + 2)}{4} \]

Test this formula with \(N = 10\), \(N = 20\), and \(N = 32\).

5.5 Simple Sequences on the UIL

Here is one where you need to do it in your head without any formula:

\[ 11 + 15 + 19 + 23 + 27 \]

First notice that if you add 11 and 17 you get 38 = 19 * 2, then note the same for 15 and 23. So what you really have is

\[ 11 + 15 + 19 + 23 + 27 = 5 \times 19 = \_\_\_\_\_\_\_\_\_\_\]

13
6 Fractions and Decimals

6.1 Comparing Fractions

If you have 2 fractions you can show that one is larger than the other by comparing the "cross product" of the two:

\[ \frac{a}{b} > \frac{f}{g} \implies a \times g > b \times f \]

Why is this true? We can show this by multiplying both sides by \( b \) and \( g \) and then cancelling terms on top and bottom that are the same:

\[ \frac{a}{b} > \frac{f}{g} \implies \frac{a \times b \times g}{b} > \frac{f \times b \times g}{g} \implies \frac{a \times g}{1} > \frac{f \times b}{1} \]

So all you need to do is to cross multiply and compare. The numerator (top) term that is smaller (or larger) is the smaller (or larger) fraction.

6.2 Examples

Which fraction is smaller: \( \frac{2}{9} \) or \( \frac{4}{17} \)?

Which fraction is bigger: \( \frac{1}{7} \) or \( \frac{3}{20} \)?

Which fraction is smaller: \( \frac{1}{11} \) or \( \frac{2}{23} \)?
6.3 Canceling Fraction Factors Before Multiplying

The UIL tests your ability to reduce problems as well as raw computing power. This means that some problems will be too hard to solve unless you see the easy way out. When it comes to fraction multiplying this important on the UIL. Let’s start by examples:

\[
\frac{5}{18} \times \frac{9}{25}
\]

This problem would be impossible to do in your head unless you realize that the 5 and 25 cancel top and bottom, as well as the 9 and 18. Once those are canceled, you have only 1 on top, and 2 * 5 on the bottom:

\[
\frac{5}{18} \times \frac{9}{25} = \frac{5 \times 9}{18 \times 25} = \frac{1}{2 \times 5} = \frac{1}{10}
\]

Here is an example of dividing fractions similar to the UIL style:

\[
3\frac{1}{4} \div \frac{1}{16} = \frac{13}{4} \div \frac{1}{16} = \frac{13}{4} \times 16 = 13 \times 4 = 52
\]

Here are some examples straight out of the UIL example test:

1. \( \frac{8}{9} \times \frac{3}{16} = \)
2. \( \frac{9}{28} \times \frac{7}{27} = \)
3. \( \frac{9}{16} \div \frac{5}{8} = \)
4. \( 6\frac{2}{3} \div \frac{1}{3} = \)
5. \( 2\frac{3}{4} \div \frac{9}{16} = \)

Don’t forget, when you divide by a fraction, you invert (flip) the dividing fraction and multiply it!
6.4 Multiplying and Dividing Decimals by 10’s

This is important to fractions because we use it next. When you multiply a decimal number by 10 you move the decimal point to the right. (Pneumonic: Multiply Right, Divide Left)

6.4.1 Examples and Exercises:

1) \( 10 \times 1.223423 = 12.23423 \)
2) \( 100 \times 1.223423 = 122.3423 \)
3) \( 1000 \times 1.223423 = . \)
4) \( 0.1212... \times 10^2 = . \)
5) \( 1.2345 \div 10 = 0.12345 \)
6) \( 1.2345 \div 100 = . \)
7) \( 1.2345 \div 1000 = . \)
8) \( 0.1111... \div 10^1 = . \)
9) \( 125 \div 10^3 = . \)
6.5 Converting Repeated Decimal to Fractions

Let's start off with an example. What is the fraction equivalent of \( .111111111 \)? The answer to this is very simple but finding it is fun. Let's use our algebra to figure this out:

\[
A = .111111111
\]
\[
\Rightarrow 10 \times A = 1.111111111 = 1 + A
\]
\[
\Rightarrow 10 \times A = 1 + A
\]
\[
\Rightarrow 10 \times A - A = 1
\]
\[
\Rightarrow (10 - 1)A = 1
\]
\[
\Rightarrow 9 \times A = 1
\]
\[
\Rightarrow A = 1/9
\]

Try to remember this because it is used many times in the next few examples.

Next is the number \( .010101..... \). We multiply by 100 this time!

\[
A = .0101010101
\]
\[
\Rightarrow 100 \times A = 1 + .0101010101 = 1 + A
\]
\[
\Rightarrow 100 \times A = 1 + A
\]
\[
\Rightarrow 100 \times A - A = 1
\]
\[
\Rightarrow (100 - 1)A = 1
\]
\[
\Rightarrow 99 \times A = 1
\]
\[
\Rightarrow A = 1/99
\]

Next is a bit trickier, but its on the UIL examples: \( .252525.... \). First notice if we divide this number by 25 we get something we know:

\[
A = .2525252525
\]
\[
\Rightarrow A/25 = .0101010101
\]
\[
\Rightarrow A/25 = 1/99
\]
\[
\Rightarrow A = 25/99
\]
Lets try one more:

\[ A = .090909... \]
\[ \Rightarrow 100 \times A = 9 + A \]
\[ \Rightarrow 99 \times A = 9 \]
\[ \Rightarrow A = 9/99 = 1/11 \]

You will want to remember these for the UIL:

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Decimal Equivalent</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{9} )</td>
<td>.111...</td>
<td>1.111.. =??</td>
</tr>
<tr>
<td>( \frac{1}{99} )</td>
<td>.010101...</td>
<td>.0202.. = 2/99(<em>explain</em>)</td>
</tr>
<tr>
<td>( \frac{1}{999} )</td>
<td>.001001001...</td>
<td>.234234 = 234 * .001001...</td>
</tr>
<tr>
<td>( \frac{1}{11} )</td>
<td>.0909...</td>
<td>.0909.. = 9 * ( \frac{1}{99} )</td>
</tr>
<tr>
<td>( \frac{1}{5} )</td>
<td>.2</td>
<td>.2 * 9 =?</td>
</tr>
</tbody>
</table>

6.5.1 Example: \( .0222 \cdots = 1/10 \times .22222 = 1/10 \times 2/9 = 2/90 \)

6.5.2 Example: \( 33.3333 \times 3 = 33\frac{1}{3} \times 3 = \frac{100}{3} \times 3 = 100 \)

6.5.3 Example: \( .1222... = .1 + .02222 = \frac{1}{10} + \frac{2}{90} = \)

6.5.4 Exercise: What is \( .123123123... \) as a fraction?

6.5.5 Exercise: What is \( .225225225... \) as a fraction?

6.5.6 Exercise: What is \( .27272727... \) as a fraction?

6.5.7 Example: What is \( 1.010101... \) as improper fraction?

6.5.8 Exercise: What is \( .11223344... \) as fraction? (Not on UIL!)
6.6 Ninths

It turns out that when you do long division to get decimal form of $1/9$ you get $0.11111...$ (Try it). Here is the ninths table

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Reduced Fraction</th>
<th>Decimal</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{9}$</td>
<td>$\frac{1}{9}$</td>
<td>$0.111..$</td>
<td>$1.111.. = ??$</td>
</tr>
<tr>
<td>$\frac{2}{9}$</td>
<td>$\frac{2}{9}$</td>
<td>$0.222..$</td>
<td>$0.0222.. = 2/90$ (explain)</td>
</tr>
<tr>
<td>$\frac{3}{9}$</td>
<td>$\frac{1}{3}$</td>
<td>$0.333..$</td>
<td>$33.333.. \times 3 = ?$</td>
</tr>
<tr>
<td>$\frac{4}{9}$</td>
<td>$\frac{4}{9}$</td>
<td>$0.444..$</td>
<td>$0.444.. \div 2 = ?$</td>
</tr>
<tr>
<td>$\frac{5}{9}$</td>
<td>$\frac{5}{9}$</td>
<td>$0.555..$</td>
<td>$0.555.. \times 9 = ?$</td>
</tr>
<tr>
<td>$\frac{6}{9}$</td>
<td>$\frac{2}{3}$</td>
<td>$0.666..$</td>
<td>$0.666.. \div 0.222.. = ?$</td>
</tr>
<tr>
<td>$\frac{7}{9}$</td>
<td>$\frac{7}{9}$</td>
<td>$0.777..$</td>
<td>$7/9 - 0.666.. = ?$</td>
</tr>
<tr>
<td>$\frac{8}{9}$</td>
<td>$\frac{8}{9}$</td>
<td>$0.888..$</td>
<td>$1 - 0.888.. = ??$</td>
</tr>
<tr>
<td>$\frac{9}{9}$</td>
<td>$\frac{1}{1}$</td>
<td>$0.999..$</td>
<td>What integer is this? explain</td>
</tr>
</tbody>
</table>

6.7 Problems and Examples

1. $\frac{6}{90} = \frac{6}{9} \times \frac{1}{10} = 0.666... \div 10 = 0.0666...
2. $\frac{13}{40} = \frac{13}{4} \times \frac{1}{10} = 3.25 \div 10 = 0.325
3. $0.444... \times 0.555... = \frac{4}{9} \times \frac{5}{9} = \frac{20}{81}
4. $0.444... \div 0.555... = \frac{4}{9} \div \frac{5}{9} = \frac{4}{5}$
We use long division to find one eight and one sixteenth:

\[
\begin{array}{c|c}
0.125 & 0.0625 \\
8 \overline{)1.000} & 16 \overline{)1.000} \\
- 80 & - 96 \\
\hline
20 & 40 \\
- 16 & - 32 \\
\hline
40 & 80 \\
- 40 & - 80 \\
\hline
0 & 0
\end{array}
\]

Now let’s summarize all the fractions that are multiples of \(1/16\) in this table. We do this by adding different fraction combinations together.

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Reduced Fraction</th>
<th>Decimal</th>
<th>How to Get it...</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{1}{16})</td>
<td>(\frac{1}{16})</td>
<td>.0625</td>
<td>Long Div</td>
</tr>
<tr>
<td>(\frac{2}{16})</td>
<td>(\frac{1}{8})</td>
<td>.125</td>
<td>1/16 + 1/16</td>
</tr>
<tr>
<td>(\frac{3}{16})</td>
<td>(\frac{3}{16})</td>
<td>.1875</td>
<td>1/8 + 1/16</td>
</tr>
<tr>
<td>(\frac{4}{16})</td>
<td>(\frac{1}{4})</td>
<td>.25</td>
<td>1/8 + 1/8</td>
</tr>
<tr>
<td>(\frac{5}{16})</td>
<td>(\frac{5}{16})</td>
<td>.3125</td>
<td>1/4 + 1/16</td>
</tr>
<tr>
<td>(\frac{6}{16})</td>
<td>(\frac{3}{8})</td>
<td>.375</td>
<td>1/4 + 1/8</td>
</tr>
<tr>
<td>(\frac{7}{16})</td>
<td>(\frac{7}{16})</td>
<td>.4375</td>
<td>3/8 + 1/16</td>
</tr>
<tr>
<td>(\frac{8}{16})</td>
<td>(\frac{1}{2})</td>
<td>.5</td>
<td>1/4 + 1/4</td>
</tr>
<tr>
<td>(\frac{9}{16})</td>
<td>(\frac{9}{16})</td>
<td>.5625</td>
<td>1/2 + 1/16</td>
</tr>
<tr>
<td>(\frac{10}{16})</td>
<td>(\frac{5}{8})</td>
<td>.625</td>
<td>1/2 + 1/8 or 10 * (\frac{1}{16})</td>
</tr>
<tr>
<td>(\frac{11}{16})</td>
<td>(\frac{11}{16})</td>
<td>.6875</td>
<td>1/2 + 3/16</td>
</tr>
<tr>
<td>(\frac{12}{16})</td>
<td>(\frac{3}{4})</td>
<td>.75</td>
<td>1/2 + 1/4</td>
</tr>
<tr>
<td>(\frac{13}{16})</td>
<td>(\frac{13}{16})</td>
<td>.8125</td>
<td>1/2 + 5/16</td>
</tr>
<tr>
<td>(\frac{14}{16})</td>
<td>(\frac{7}{8})</td>
<td>.875</td>
<td>1/2 + 3/8</td>
</tr>
<tr>
<td>(\frac{15}{16})</td>
<td>(\frac{15}{16})</td>
<td>.9375</td>
<td>1/2 + 7/16</td>
</tr>
<tr>
<td>(\frac{16}{16})</td>
<td>1</td>
<td>1.0</td>
<td>Any way you like!</td>
</tr>
</tbody>
</table>
7 Algebra Tricks and Treats!

Who says algebra is boring? We’ll most people just don’t understand that algebra is everywhere, even in your breakfast serial ;). Companies use algebra to create formulas for food recipes, cars, airplanes, electricity and even water works! Let’s go over some of the easy tricks that you will see in the UIL.

7.1 Plus One - Minus One

This trick is based on the identity formula:

\[(A + 1) \times (A - 1) = A^2 - 1\]

The UIL will ask you to multiply two big numbers that differ by 2 expecting you to recognize that formula. Let’s try the example 701*699. Normally you would have to break out a page of paper or a calculator to do this one, but he formula above gives us an easy way:

\[
201 \times 199 = 200^2 - 1 = 40000 - 1 = 39999 \\
119 \times 121 = 120^2 - 1 = 14400 - 1 = 14399 \quad (Hint: 120^2 = 12 \times 12 \times 100) \\
701 \times 699 = 700^2 - 1 = 490000 - 1 = 489999
\]

7.1.1 Exercises

1. 99 * 101 = 
2. 109 * 111 = 
3. 299 * 301 = 
4. 1001 * 999 = 
5. 119 * 121 =
7.2 More Algebra: Difference of Squares

This trick is based on the formula:

\[ a^2 - b^2 = (a + b) \times (a - b) \]

The UIL will have a problem like this: \(55^2 - 45^2\). Normally you would have to write a few things down to get this but with the formula you get it fast:

\[ 55^2 - 45^2 = (55 + 45) \times (55 - 45) = 100 \times 10 = 1000 \]

Try these exercises:

\[ 51^2 - 49^2 = \]
\[ 41^2 - 39^2 = \]
\[ 32^2 - 28^2 = \]
\[ 28^2 - 22^2 = \]
\[ 21^2 - 11^2 = \]
\[ 101^2 - 99^2 = \]
\[ 25^2 - 24^2 = \]
\[ 53 \times 47 = \] (hint: use the rule backwards)
7.3 More Algebra: Solving Equations

The UIL gives some small algebra problems involving equations and inequalities. Both equations and inequalities are solved in a similar way.

The golden rule is:
Always do same (operation) to both sides of any equation!

Example: Solve the $5 + 3x = 35$ for $x$:

\[
5 + 3x = 35 \\
3x = 30 \quad \text{(subtracted 5 from both sides)} \\
x = 10 \quad \text{(Divide both sides by 10)}
\]

Example: Solve the inequality $5 - 3x > 35$ for $x$:

\[
5 - 3x > 35 \\
5 > 3x + 35 \quad \text{(added 3x to both sides)} \\
-30 > 3x \quad \text{(subtracted 35 from both sides)} \\
-10 > x \quad \text{(Divide both sides by 10)}
\]

Solve the following examples:

Ex 7.36: $22A - 1 = 10$
Ex 7.37: $3A - 27 = 0$
Ex 7.38: $3A - 24 = A$
Ex 7.39: $5Y + 12 = 32$
Ex 7.40: $4 - A = 5$
Ex 7.41: $7A + 5 = 5$
Ex 7.42: $12 + A = 5A$
8 Sets

The UIL covers basic set theory of unions and intersections.

8.1 Unions

The union of two sets is the combination of both sets. You don’t count the repeated elements more than once:

The union symbol is $\cup$.

Here is an example:

$$\{T, E, X, A, S\} \cup \{M, A, T, H\} = \{T, E, X, A, S, M, H\}$$

You don’t write the A and T twice here. Try this example:

$$\{A, B, C, D, E\} \cup \{D, E, F, G\} =$$

8.2 Intersections

The intersection of 2 sets is the set of all common elements in each set. Its also called the overlap of the two sets. In other words you just write the elements that both sets contain. Example:

$$\{T, E, X, A, S\} \cap \{M, A, T, H\} = \{A, T\}$$
$$\{S, U, M, M, I, T, T\} \cap \{E, A, G, L, E, S\} = \{S\}$$
$$\{M, A, T, H\} \cap \{L, A, N, G, U, A, G, E\} = \{A\}$$

You don’t want to include any repeated elements in any intersection or union.
9 Primes, Factors, GCF, LCM

The UIL tests have a few prime number questions. They expect you to memorize some of the primes and know a few ideas about primes. Here are the prime numbers from 1 to 61. Try to memorize at least the first 10 of these:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61

9.1 Twin Primes

Twin Primes are primes that differ only by 2. For example these pairs are twin primes:

{2, 3}, {3, 5}, {5, 7}, {11, 13}, {17, 19}, {29, 31}, {41, 43}

Fun fact: the largest known twin primes are $2003663613 \cdot 2^{195000} \pm 1$ and both have 58711 digits! There could be others, you never know ☺.

Ex 9.43: What is the twin prime of 59?
Ex 9.44: What is the twin prime of 73?
Ex 9.45: What is the twin prime of 101?
9.2 GCF and LCM

The GCF and LCM of 2 or more numbers are closely related to prime numbers they contain.

The **GCF** is the **Greatest Common Factor** that both numbers contain. Its less than or equal to either number. For example, the GCF of 2 and 3 is 1, because they contain nothing in common. The GCF of 2 and 4 is 2. The GCF of 4 and 6 is 2.

Find the factors of each number and pick the largest factor that each number shares, that is the GCF.

9.3 Exercises

1. What is the GCF of 16 and 24?
2. What is the GCF of 12 and 18?
3. What is the GCF of 22 and 33?
4. What is the GCF of 24 and 36?

The **LCM** is the **Lease Common Multiple** of both. Its at least as big as the biggest number. Sometimes it is the biggest number. For example the LCM of 4 and 16 is 16. The LCM of 2 and 3 is 6 though.

If you look at the multiples of 2 and 3 again we have:

\[ 2, 4, 6, 8, 10, \ldots \quad 3, 6, 9, 12, \ldots \]

You see that 6 is the smallest number that comes up. That is our answer. Lets try 6 and 8

\[ 6, 12, 18, 24, \ldots \quad 8, 16, 24, 32, \ldots \]

What is the smallest number that those sequences share? Right, 24.
But how are these related to the primes? You must look at the prime factorization of the numbers to see it. Let’s look at 18 and 24, and write them as prime factors with a power:

\[
\begin{array}{ccccccccccc}
18 &=& 2^1 & 3^2 & 5^0 & 7^0 & 11^0 & 13^0 & 17^0 & 19^0 \\
24 &=& 2^3 & 3^1 & 5^0 & 7^0 & 11^0 & 13^0 & 17^0 & 19^0
\end{array}
\]

The LCM is 72, and the GCF is 6, so prime-factor them the same way:

\[
\begin{array}{ccccccccccc}
\text{LCM} &=& 2^3 & 3^2 & 5^0 & 7^0 & 11^0 & 13^0 & 17^0 & 19^0 \\
\text{GCF} &=& 2^1 & 3^1 & 5^0 & 7^0 & 11^0 & 13^0 & 17^0 & 19^0
\end{array}
\]

It turns out that the LCM is the prime factorization of the two number using the \textit{Largest} of each power and the GCF is the prime factorization using the \textit{Smallest} of each power. This method works for multiple numbers at a time too:

\[
\begin{array}{ccccccccccc}
18 &=& 2^1 & 3^2 & 5^0 & 7^0 & 11^0 & 13^0 & 17^0 & 19^0 \\
24 &=& 2^3 & 3^1 & 5^0 & 7^0 & 11^0 & 13^0 & 17^0 & 19^0 \\
36 &=& 2^2 & 3^2 & 5^0 & 7^0 & 11^0 & 13^0 & 17^0 & 19^0 \\
22 &=& 2^1 & 3^0 & 5^0 & 7^0 & 11^1 & 13^0 & 17^0 & 19^0 \\
\text{LCM} &=& 2^3 & 3^2 & 5^0 & 7^0 & 11^1 & 13^0 & 17^0 & 19^0 \\
\text{GCF} &=& 2^1 & 3^0 & 5^0 & 7^0 & 11^0 & 13^0 & 17^0 & 19^0
\end{array}
\]

So the \textit{LCM} = 792 and \textit{GCF} = 2 for this example.
This means that we have for LCM and GCF:

\[
\begin{array}{cccccccccc}
\text{LCM} = & 2^3 & 3^2 & 5^0 & 7^0 & 11^1 & 13^0 & 17^0 & 19^0 \\
\text{GCF} = & 2^1 & 3^0 & 5^0 & 7^0 & 11^0 & 13^0 & 17^0 & 19^0
\end{array}
\]

So the LCM = 36 * 11 = 396 and the GCF = 6 Of course the UIL doesn’t allow you to write all this stuff down, so you have to use the manual methods for the GCF and the shortcut we talk about next:

9.4 GCF and LCM Shortcut

If you think about what we just did, you will come to the fact that the multiple of the GCM and LCF of two numbers (A and B) is the product of the two numbers:

\[
GCF \times LCM = A \times B
\]

Can you prove this to your classmates?

This makes it a lot easier to figure out the LCM or GCF if you know one or the other. So for example the LCM of 12 and 40 is:

\[
LCM(12, 40) = 12 \times 40 / GCF = 12 \times 40 / 4 = 12 \times 10 = 120
\]

Remember: Don’t multiply anything until you have cancelled all you can cancel. This speeds up your answer and leads less mistakes! Here are some exercises to try:

\text{Ex 9.46:} \quad LCM(12, 50) = 12 \times 50 / GCF
\text{Ex 9.47:} \quad LCM(14, 40) =
\text{Ex 9.48:} \quad LCM(14, 49) =
\text{Ex 9.49:} \quad LCM(15, 50) =
\text{Ex 9.50:} \quad LCM(25, 60) =
Roman Numerals are formed by combining symbols together and adding the values. Generally, symbols are placed in order of value, starting with the largest values. Repeated numerals are added. When smaller values precede larger values, the smaller values are subtracted from the larger values, and the result is added to the total. For example, MMVI = 1000 + 1000 + 5 + 1 = 2006.

<table>
<thead>
<tr>
<th>Roman</th>
<th>Arabic</th>
<th>Possible Origin</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>1</td>
<td>Scratch on a Stick or Stone</td>
</tr>
<tr>
<td>V</td>
<td>5</td>
<td>Double Scratch or Notch</td>
</tr>
<tr>
<td>X</td>
<td>10</td>
<td>Cross Cut</td>
</tr>
<tr>
<td>L</td>
<td>50</td>
<td>Inverted T</td>
</tr>
<tr>
<td>C</td>
<td>100</td>
<td>centum (Latin)</td>
</tr>
<tr>
<td>D</td>
<td>500</td>
<td>Inverted C with V on it</td>
</tr>
<tr>
<td>M</td>
<td>1,000</td>
<td>mille (Latin)</td>
</tr>
</tbody>
</table>

So 2012 in Roman numerals is MMXII. Thus CXXI is 121. Here are some exercises to practice:

**Ex 10.51:** MX + MI = (Arabic numeral)

**Ex 10.52:** IX + XI = (Arabic numeral)

**Ex 10.53:** MMXIII = (Arabic numeral)

**Ex 10.54:** CMXIV = (Arabic numeral)

**Ex 10.55:** MCMXLIV = (Arabic numeral)
The UIL also has problems that involve other number bases like base 2. This means that digits only are 0 and 1 in base 2 for example. To understand bases, we need to talk about powers again. In base 10 for example, we know that the one’s only go from 0 to 9. The next number puts a single 1 in the 10’s column.

<table>
<thead>
<tr>
<th>Place</th>
<th>Power</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000’s</td>
<td>10^3</td>
</tr>
<tr>
<td>100’s</td>
<td>10^2</td>
</tr>
<tr>
<td>10’s</td>
<td>10^1</td>
</tr>
<tr>
<td>1’s</td>
<td>10^0</td>
</tr>
<tr>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>1/10</td>
<td>10^-1</td>
</tr>
<tr>
<td>1/100</td>
<td>10^-2</td>
</tr>
<tr>
<td>1/1000</td>
<td>10^-3</td>
</tr>
<tr>
<td>1/10000</td>
<td>10^-4</td>
</tr>
</tbody>
</table>

11.1 Base 2

In base 2, you only have 2 digits to work with: 0 and 1. That means every number is made of zeros and ones only!

<table>
<thead>
<tr>
<th>Place</th>
<th>Power</th>
</tr>
</thead>
<tbody>
<tr>
<td>8’s</td>
<td>2^3</td>
</tr>
<tr>
<td>4’s</td>
<td>2^2</td>
</tr>
<tr>
<td>2’s</td>
<td>2^1</td>
</tr>
<tr>
<td>1’s</td>
<td>2^0</td>
</tr>
<tr>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>1/2</td>
<td>2^-1</td>
</tr>
<tr>
<td>1/4</td>
<td>2^-2</td>
</tr>
<tr>
<td>1/8</td>
<td>2^-3</td>
</tr>
<tr>
<td>1/16</td>
<td>2^-4</td>
</tr>
</tbody>
</table>

So how do you figure out a number in base 2? Let’s try some easy numbers:

7 = 4 + 2 + 1 = 111_2 (the subscript 2 says the number is base 2)
9 = 8 + 0 + 0 + 1 = 1001_2 (no 2’s and no 4’s)
3 = 2 + 1 = 11_2 (one 2’s and one 1)
So 7 has one \(2^2\), one \(2^1\), and one \(2^0\). But what about really big numbers? How do you figure out those hard ones? It turns out there is an easy recipe (algorithm) for figuring out any size number in any base! It works like this:

- Divide the number (base 10) by your base (2 in this example), store answer, write down remainder in ones place.
- Divide previous result again by base, store answer and write down remainder in next column (in the 2’s place).
- Repeat last step and stop when you get to zero.

Let’s try it on 7 base 2 again to see how it works:

- Divide 7 by 2, keep 3 (on your fingers or head), write down the remainder 1 (ones place).
- Divide 3 by 2, keep 1 (on your fingers or head), and write 1 (in the 2’s place).
- Divide 1 by 2, (zero), and write remainder 1 (4’s place).
- Stop (nothing left). The answer in base 2 is \(111_2\).
11.2 Base 3

In base 3, you only have 3 digits to work with: 0, 1, and 2.

<table>
<thead>
<tr>
<th>27’s</th>
<th>9’s</th>
<th>3’s</th>
<th>1’s</th>
<th>.</th>
</tr>
</thead>
<tbody>
<tr>
<td>3³</td>
<td>3²</td>
<td>3¹</td>
<td>3⁰</td>
<td>.</td>
</tr>
</tbody>
</table>

| 1/3 | 1/9 | 1/27 | 1/81 |

Place 27’s 9’s 3’s 1’s .

Power 3⁻¹ 3⁻² 3⁻³ 3⁻⁴

Let’s try some easy numbers again in base 3:
2 = 2 = 2₃  (same as in base 10)
3 = 3 + 0 = 10₃  (one 3 and zero 1’s)
5 = 3 + 2 = 12₃  (one 3 and two 1’s)
7 = 2 * 3¹ + 1 = 21₃  (two 3’s and one 1)
12 = 1 * 3² + 1 * 3¹ + 0 * 3⁰ = 110₃  (two 3’s and one 1)

Let’s try the recipe (algorithm) we talked earlier on 13. This time we divide each step by 3 and write the remainder, keeping the result for the next step:

- Divide 13 by 3, keep 4, write down the remainder 1 (in ones place).
- Divide 4 by 3, keep 1, and write 1 in the 3’s place.
- Divide 1 by 3, (zero), and write remainder 1 in the 9’s place.
- Stop (nothing left). The answer is 111₃.

11.3 Base 4

In base 4, you have the digits (0, 1, 2, 3). Let’s just go straight into base 4 calculations. In base four you have 1’s, 4’s, 16’s, 64’s, which corresponds to (4⁰, 4¹, 4², 4³, ....)

5 = 1 * 4¹ + 1 = 11₄  (one 4’s and one 1’s)
7 = 1 * 4¹ + 3 = 13₄  (one 4’s and three 1’s)
12 = 3 * 4¹ + 0 = 30₄  (three 4’s and zero 1’s)
11.4 Adding Strange Bases

To add other bases you just need to remember what digits you can have and carry over the extra digit to the next place. For example in base 4:

\[ 11_4 + 10_4 = 21_4 \quad (\text{ie. } 5 + 4 = 9) \]

11.5 Exercises

Add the following numbers in bases given:

Ex 11.56: \[ 1_2 + 1_2 = \]
Ex 11.57: \[ 101_2 + 1_2 = \]
Ex 11.58: \[ 110_2 + 100_2 = \]
Ex 11.59: \[ 111_3 + 111_3 = \]
Ex 11.60: \[ 121_3 + 121_3 = \]
Ex 11.61: \[ 321_4 + 123_4 = \]
Ex 11.62: \[ 132_4 + 212_4 = \]

\[
\begin{array}{c}
22_4 \\
+ 13_4 \\
\hline
12_4 \\
\end{array}
\]

Ex 11.63: \[ + 13_4 \]

\[
\begin{array}{c}
22_4 \\
- 13_4 \\
\hline
10_4 \\
\end{array}
\]

Ex 11.64: \[ + 3_4 \]

\[
\begin{array}{c}
101_2 \\
+ 11_2 \\
\hline
21_4 \\
\end{array}
\]

Ex 11.65: \[ + 23_4 \]